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On some aspects of survival under production uncertainty*

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I. Introduction

In recent years there has emerged a considerable volume of literature which analyzes the possibility of a certain performance index staying above a minimum level over time for an agent who operates in a given stochastic dynamic environment. Thus, one may be interested in the potential ability of an agent to meet a minimum positive "subsistence" consumption level over time or of firms being able to pay out a minimum level of dividend to shareholders or to meet a minimal level of debt service in order to avoid bankruptcy. For a renewable resource which is exploited over time at a rate determined by the market or by the optimal policy of its owner, one can examine the circumstances under which the resource does not become extinct over time.

The literature on exhaustible resources has focused on the difficulties of maintaining a positive steady level of consumption in an economy which relies on an exhaustible resource as an essential input in production [see Solow (1974), Cass and Mitra (1991)]. In a model of renewable resource with stochastic and concave production function, where the resource is depleted every time period according to market equilibrium, Mirman and Spulber (1984) derived results on chances of survival for the resource.

Majumdar and Radner (1991, 1992) consider a dynamic model of consumption and investment with production uncertainty, where the agent is required to meet a certain strictly positive consumption level every time period in order to survive. They consider cases where the production function is of the canonical neoclassical type and also the case where it is linear. This framework is close to the classical gambler's ruin problem in probability theory (see Feller (1957), Billingsley (1979)). In a similar framework, Ray (1984) analyzed the survival problem of economic agents who had the option of borrowing and lending over time.

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There is also an expanding literature, both theoretical and experimental, on decision making under bounded rationality where agents attempt to maximize their chances of attaining some "aspiration level" or some minimum level of some performance index through their actions [see Simon (1955), Radner (1975), Gordon (1985), Karni and Schmeidler (1986), Majumdar and Radner (1991); also see the surveys by March (1988) and Selten (1990)].

This paper is not specifically concerned with the optimization problem of agents. Our framework is the same as that of Majumdar and Radner (1991, 1992). The agent is required to meet a minimum subsistence consumption level over time. Given a specific technology which transforms investment into output or returns over time and a sequence of random shocks which affect these returns, we are interested in the *maximum potential probability that the agent is able to meet an exogenously specified subsistence consumption requirement every time period.* Our purpose is to characterize the qualitative properties of this "survival" probability as a function of the initial endowment of the agent and other parameters of the model. We note that Majumdar and Radner (1991) obtain an explicit formula for the survival probability function in a continuous-time model. However, in discrete-time models results on the nature of the survival probability function are rather limited.

In order to analyze the potential possibility of attaining a subsistence consumption level every time period we focus on the stochastic process of input into production over time, assuming that the agent fixes his consumption in every time period at the subsistence level $c > 0$. The agent is endowed with an initial stock $x > 0$ in period zero. In period 1, the agent receives output $h(x, \theta_1)$ where θ_1 is the random production shock in period 1 and h is the (stationary) return function. The agent is ruined in period 1 if the output received is not greater than c. Otherwise, he consumes c and uses $X_1 = h(x, \theta_1) - c$ as input into production of next period's output $h(X_1, \theta_2)$ and the process is repeated. The agent is ruined in the first period T for which the available output $h(X_{T-1}, \theta_T) \leq c$; that is, $X_T \leq 0$. We define survival as the event that $X_t > 0$ for all $t \geq 1$. We take $\{\theta_t\}_{t=1}^{\infty}$ to be a sequence of independent and identically distributed random variables whose support is a closed interval in the positive real line and whose common distribution has a smooth density.

In Sections II–IV we characterize the probability of survival and develop a functional equation for it, under a set of fairly general assumptions on the stationary return function h. In particular, these assumptions allow as special cases the "bounded growth" and the "linear" technology of Majumdar and Radner (1991, 1992) as well as the "productive" technology considered by Gale and Sutherland (1968). In Section II, we establish the existence of two critical points α and β, for any given level of c, such that ruin is inevitable if the initial stock x is no greater than α and survival occurs almost surely if x is no less than β. Let $V(x)$ denote the probability of survival from initial stock x, given c. The focus of attention is the behavior of the (survival probability) function V on the interval $[\alpha, \beta]$. We show that V is strictly increasing on $[\alpha, \beta]$.

In Section III, we establish a functional equation for the survival probability function, V. Like the optimality equation of dynamic programming, this functional equation is based on a recursive logic. However, it is not the outcome of any optimization exercise. In Section IV we establish a set of smoothness properties of the survival probability function, V, using the functional equation. In particular we prove the twice continuous differentiability of V on $[\alpha, \beta]$. This allows us to investigate the behavior of the derivatives of V of various orders, and convert the functional equation, which is really an integral equation, to differential equations of various orders.

The twice continuous differentiability of V on $[\alpha, \beta]$ and the fact that the slopes are zero at the endpoints implies that the second derivative of V cannot have the same sign on the entire interval. However, V can be shown to be concave at the upper end of the interval $[\alpha, \beta]$. Thus, if (the graph of) V has any "regular" shape, it must be *S-shaped* on $[\alpha, \beta]$. In Section V, we establish the S-shape of the survival probability function in some specific cases assuming that the θ_t's are uniformly distributed. In Section Va, we consider the case where $h(x, \theta)$ is of the form θx. In Section V, we consider the case where $h(x, \theta))$ is of the form $\theta f(x)$ and f is strictly concave on $[\alpha, \beta]$.

In Section VI, we try to bring out the implications of the S-shape of the survival probability function by considering an example of aid allocation. An amount $b > 0$ is to be distributed as aid to n agents with initial wealth distributed over the $[\alpha, \beta]$ interval. The objective function is to maximize the sum of increments in probability of survival of the n-agents as a result of aid. It is shown that an effective aid policy is not necessarily egalitarian. The optimal rule for low levels of b is not to give "the greatest aid to the poorest" but rather to give priority to agents with high levels of V' typically located in the "middle" region.

In Section VII, we consider the effects of a change in the distribution of the random shocks. The probability of survival is shown to be nondecreasing in first order improvements in the distribution of the shocks. In a simple example, we show that for a certain "low" range of initial wealth, a more variable distribution of shocks in the sense of mean-preserving spread, ensures a higher probability of survival; the opposite holds for a certain "high" range. This is in line with results on "risk loving" behavior by agents at low levels of wealth in portfolio choice with survival as objective as noted in Majumdar and Radner (1991).

II. Preliminaries

The model we consider is characterized by an initial stock $x > 0$, a constant "subsistence" consumption $c > 0$, a sequence of random variables $\{\theta_t\}_{t=1}^{\infty}$ and a stationary return (production) function $h(y, \theta)$ where y is the input level and θ is a realization of the random variable θ_t.

We make the following assumptions on $\{\theta_t\}_{t=1}^{\infty}$:

(A1) *The θ_t's are independent and identically distributed random variables defined on some probability space (Ω, F, μ), with support $[\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} < \bar{\theta} < \infty$.*

(A2) *The distribution of the θ_t's has a density $g(\theta)$ which is continuously differentiable on $[\underline{\theta}, \bar{\theta}]$.*

The return function $h(\bullet, \bullet)$ is a mapping from $(\Re_+) \times [\underline{\theta}, \bar{\theta}]$ into $(\Re_+)$. We make the following assumptions on h:

(A3) *For any $x > 0$, θ_1, $\theta_2 \in [\underline{\theta}, \bar{\theta}]$, $\theta_1 < \theta_2$ implies $h(x, \theta_1) < h(x, \theta_2)$. For any $\theta \in [\underline{\theta}, \bar{\theta}]$, $0 \leq x_1 < x_2$ implies $h(x_1, \theta) \leq h(x_2, \theta)$.*

(A4) *For any* $\theta \in [\underline{\theta}, \bar{\theta}]$, $h(0, \theta) = 0$.

(A5) *For any given* $\theta \in [\underline{\theta}, \bar{\theta}]$, $h(\bullet, \theta)$ *is continuous on* $\Re_+$ *and thrice continuously differentiable on* $\Re_{++}$. *For any* $y > 0$, $h(y, \bullet)$ *is thrice continuously differentiable on* $[\underline{\theta}, \bar{\theta}]$.

The next assumption is made in order to ensure that the maximum sustainable consumption from the technology $h(\bullet, \theta)$ is not less than the "subsistence" constant level of consumption c, whatever be θ.

(A6) *For any* $\theta \in [\underline{\theta}, \bar{\theta}]$, *there exists* $x(\theta) > 0$ *such that* $[h(x(\theta), \theta) - c] = x(\theta)$ *and* $[h(x, \theta) - c] < x$ *for all* $0 \leq x < x(\theta)$.

Assumption (A6) implies that for each θ, the function $[h(\bullet, \theta) - c]$ intersects the 45° line for the first time at $x(\theta)$. It is easy to see that $\theta_1 < \theta_2$ implies $x(\theta_1) > x(\theta_2)$. Denoting $x(\bar{\theta})$ by α and $x(\underline{\theta})$ by β, we observe that (A6), along with (A3), ensures that for any $\theta \in (\underline{\theta}, \bar{\theta}]$, the function $[h(\bullet, \theta) - c]$ after intersecting the 45° line at $x(\theta)$ stays above the 45° line till $x(\underline{\theta}) = \beta$.

It is worth noting that the assumptions (A3)–(A6) allow for a wide class of production functions provided the maximum sustainable consumption for such technologies are no less than c. In particular, note that no restrictions on the magnitude of the slopes or curvature of the production function is imposed.

Before we go to the analysis of the model, let us extend the domain of $h(\bullet, \theta)$ to the entire real line by defining for any $\theta \in [\underline{\theta}, \bar{\theta}]$, $h(x, \theta) = 0$ for all $x \leq 0$.

Consider an economic agent who consumes a constant amount $c > 0$ every period and is ruined if the output available for consumption falls below that level. The agent starts off with a stock x in period zero. This stock is transformed into output in period 1 through the production function $h(x, \theta_1)$ where θ_1 is the random shock introducing uncertainty in the production process of period 1. If $h(x, \theta_1)$ is no greater than c, then the agent is ruined in period 1. If $h(x, \theta_1)$ is larger than c, then the agent consumes c and carries over $X_1 = [h(x, \theta_1) - c]$ as input into the production process of period 2. The output in period 2 is $h(X_1, \theta_2)$ where θ_2 is, again, a random shock. The agent is ruined in period 2 if $h(X_1, \theta_2)$ is no greater than c. If $h(X_1, \theta_2) > c$ then the agent consumes c and carries over $X_2 = [h(X_1, \theta_2) - c]$ as input into the production process of period 3 and so on. In general, the agent is ruined in the first (random) period T for which $h(X_{T-1}, \theta_T) \leq c$; that is, $X_T \leq 0$. The agent is said to *survive* (forever) if $X_t > 0$ for all $t \geq 1$.

More formally, let $X_t(x)$ be the stock at the end of time period t, given that the initial stock is x and given a fixed consumption level c. The process $\{X_t(x)\}$ can be written as:

$$X_0(x) = x, X_t(x) = h(X_{t-1}(x), \theta_t) - c, \quad \text{for } t \geq 1 \tag{1}$$

Let $V(x)$ denote the probability of survival from initial stock x, for a fixed consumption level c. That is,

$$V(x) = \text{Probability } [X_t(x) > 0, t \geq 1] \tag{2}$$

We refer to $V: \Re \to [0, 1]$ as the *survival probability function*. It is obvious that $V(x) = 0$ for $x \leq 0$.

We now proceed to state two results (with some heuristic arguments) which are basically extensions of the results derived in Majumdar and Radner (1992) and Mirman and Spulber (1984) for the "bounded growth" case to the general model considered here. [For rigorous proofs see Mitra and Roy (1990a).]

Proposition 1:
The survival probability function, V, satisfies the following properties: (i) $V(x) = 0$ *for* $x \leq \alpha$; (ii) $0 < V(x) < 1$ *for* $\alpha < x < \beta$; (iii) $V(x) = 1$ *for* $x \geq \beta$.

The result is intuitively clear from Figure 1. If the initial stock is below α, then even if the best possible outcome of θ_t (that is, $\bar{\theta}$) is realized period after period, the agent's stock can only decrease over time and in fact falls below zero in a finite number of time periods. Thus ruin occurs almost surely. On the other hand if the initial stock is above β, then even under the worst possible realizations of θ_t (that is, $\underline{\theta}$), period after period, the agent's stock can never fall below β. Hence the agent survives almost surely.

Now, suppose the initial stock (z_1) lies between α and β. Consider the range of θ's for which $x(\theta)$ lies between α and z_1; that is $[h(\bullet, \theta) - c]$ intersects the 45° line at a point below S. Now, if the θ_t's successively lie in this range then in a finite number of time periods, the agent's stock is driven up to a level above β from which survival is ensured. Thus the probability of survival from x, is at least as much as the measure of the set of $\omega \in \Omega$ for which θ_t's lie in the specified range for the appropriate finite number of time periods and this measure is positive; that is, $V(z_1) > 0$. Similarly, if the θ_t's lie in the range of θ's for which $x(\theta)$ lies between z_1 and β, then the agent is ruined in a finite number of time periods. Since the probability of this event is positive, the probability of ruin from z_1 is positive. Consequently, the probability of survival from z_1 is less than 1; that is, $V(z_1) < 1$.

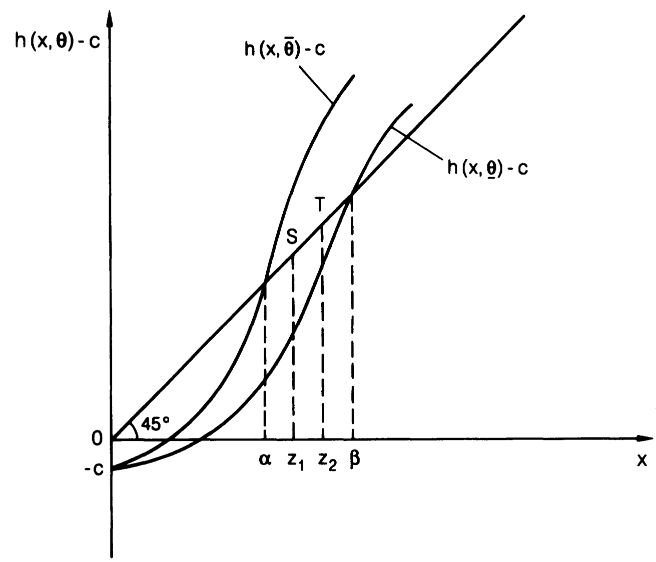

Figure 1.

Proposition 2:
The survival probability function, V, is strictly increasing on $[\alpha, \beta]$.

That $V(x)$ is non-decreasing is obvious. To see why $V(x)$ is strictly increasing consider $z_1 < z_2$ as in Figure 1. If the θ_t's lie successively in the range of θ's for which $x(\theta) \in (z_1, z_2)$ that is, $[h(\bullet, \theta) - c]$ intersects the 45° line at points lying between S and T, then there exists a finite number of time periods τ in which the stock from the process beginning at z_1 decreases to a level below α from where ruin is inevitable and that from z_2 increases to a level above β from where survival is ensured. Hence the probability of survival from z_1 and z_2 differ by at least as much as the measure of the set of $\omega \in \Omega$ for which θ_t's lie in the prescribed range for τ periods and that measure is positive. Thus, $V(z_1) < V(z_2)$.

III. The functional equation of the probability of survival

Consider the probability of survival from a given initial stock $x > 0$. Starting from x the stock at the end of next period is $[h(x, \theta_1) - c]$. Now, suppose the distribution of the θ_t's collapsed on two points – say $\hat{\theta}$ and $\tilde{\theta}$ with probability p and $(1-p)$ respectively. Then the event that the agent survives (forever) from $x > 0$ is equal to the event that survival occurs from at least one of the two possible levels of stock for the next time period; that is, from $[h(x, \hat{\theta}) - c]$ and $[h(x, \tilde{\theta}) - c]$. Therefore, in such a case, one should have:

$$V(x) = pV(h(x, \hat{\theta}) - c) + (1-p)V(h(x, \tilde{\theta}) - c)$$

If we extend this heuristic argument to the case where the θ_t's have a density $g(\theta)$ on $[\underline{\theta}, \bar{\theta}]$, as has been assumed in section II, then we expect that the following relation should hold:

$$V(x) = E[V(h(x, \theta) - c)]$$

This is the functional equation of the probability of survival. It can be derived as a standard property of Markov processes; for an elementary self-contained proof see Mitra and Roy (1990a). For ease of reference, we state this result formally in the following theorem:

Theorem 1:
For every $x > 0$, *the survival probability function, V, satisfies the following equation*:

$$V(x) = \int_{\underline{\theta}}^{\bar{\theta}} V(h(x, \theta) - c) g(\theta) d\theta. \tag{3}$$

Remark:
Note that since V is increasing, it is almost everywhere continuous. Thus the integral on the right hand side of the functional equation (3) can be treated as a Riemann Integral (see Rudin (1976) pp. 323).

It is very difficult, in general, to obtain a closed form solution to the functional equation (3). In fact, this is one of the well known problems faced by probabilists working on the gambler's ruin or random walk theory. The best one can hope for, under the circumstances, is to characterize the qualitative properties of the function

$V(x)$ by using the functional equation. In sections (IV)–(VI) we work out some of the interesting properties of $V(x)$ under various structural assumptions.

IV. Smoothness properties of the survival probability function

In this section we develop certain continuity and differentiability properties of the survival probability function. While such "smoothness" properties of the function are of independent interest as qualitative characterizations, they also enable us to investigate the shape of the function by looking at derivatives and differential equations of various orders.

To begin, note that we have already specified the function V explicitly everywhere on $\mathfrak{R}$ except the interval $[\alpha, \beta]$ (see Proposition 1). We also know that $V(x)$ is strictly increasing on $[\alpha, \beta]$. Denote by $H(x, s)$ the "partial inverse" of $h(x, \theta)$; that is, $H(x, s) = \{\theta : h(x, \theta) = s\}$ for $s \in [h(x, \underline{\theta}), h(x, \bar{\theta})]$. Then, the functional equation (3) can be written, after a change of variables, as

$$V(x) = \int_{h(x,\underline{\theta})-c}^{h(x,\bar{\theta})-c} V(t) g(H(x, t+c)) D_2 h(x, H(x, t+c))\, dt \tag{4}$$

where $D_2 h$ is the partial derivative of h with respect to the second argument.

Define $A = h(\alpha, \underline{\theta}) - c$ and $B = h(\beta, \bar{\theta}) - c$. Then, $A \leq h(x, \underline{\theta}) - c < h(x, \bar{\theta}) - c \leq B$ for all $x \in [\alpha, \beta]$.

Choose any $x \in (\alpha, \beta)$ and any sequence $\{x_n\}$ converging to x. Then, there exists N, such that for $n \geq N$, $x_n \in [\alpha, \beta]$; that is, $A \leq h(x_n, \underline{\theta}) - c < h(x_n, \bar{\theta}) - c \leq B$. Note that $x_n \to x$ implies, $H(x_n, t+c) \to H(x, t+c)$ pointwise. Also, g, $D_2 h$ and h are continuous functions. Hence, writing out the functional equation of the form (4) for $V(x_n)$ and taking limits using the dominated convergence theorem, we have $V(x_n) \to V(x)$. This proves continuity of V on (α, β).

It is easy to show that V is also continuous at α and β. For example, at α the left hand limit is equal to zero which is equal to $V(\alpha)$. Consider any sequence $x_n \downarrow \alpha$. Then, by following a method similar to that in the previous paragraph, we have $V(x_n) \downarrow V(\alpha)$ so that the right hand limit is equal to $V(\alpha)$.

Let D be the rectangle in $\mathfrak{R}^2$ defined by $D = [A, B] \times [\underline{\theta}, \bar{\theta}]$. Define

$$F(t, x) = V(t) g(H(x, t+c)) D_2 h(x, H(x, t+c)) \tag{5}$$

Then F and $D_2 F$ are continuous on D. Furthermore, $[h(x, \bar{\theta}) - c]$ and $[h(x, \underline{\theta}) - c]$ lie in $[A, B]$ and are differentiable on $[\alpha, \beta]$. Thus by an appeal to the "Leibniz Formula" (see Bartle (1976), p. 245) we have from (4) that V is differentiable on $[\alpha, \beta]$. In fact, essentially repeating this argument, one can show that V is twice continuously differentiable on $[\alpha, \beta]$ and $V'(\alpha) = V'(\beta) = 0$. [For a complete proof see Mitra and Roy (1990a)]. The above results can be formally summarized in the following theorem:

Theorem 2:
The survival probability function V, is twice continuously differentiable on $[\alpha, \beta]$ *and*

$$V'(\alpha) = V'(\beta) = 0. \tag{6}$$

V. The survival probability function is S-shaped

In this section, we develop some results on the "convexity-concavity" properties of $V(x)$ in the range $[\alpha, \beta]$. In particular, we prove that $V(x)$ is S-shaped for two classes of technologies, a linear and a "canonical" nonlinear case. We assume for this purpose (in addition to (A.1) and (A.2)) that $g(\theta)$ has a very simple form: it is the uniform density on $[\underline{\theta}, \bar{\theta}]$.

V.a: The case of a linear technology with uniformly distributed shocks

In this subsection we consider the special case where $h(x,\theta)$ is *linear*; that is $h(x,\theta) = x\theta$, for $\theta \in [\underline{\theta}, \bar{\theta}]$, $x \geq 0$ and $h(x,\theta) = 0$ for $\theta \in [\underline{\theta}, \bar{\theta}]$ and $x \leq 0$. Thus, (A.3)–(A.5) are satisfied. Assumption (A.6) is ensured by assuming that the linear technology is productive ($\underline{\theta} > 1$).

For this linear case, we can actually calculate what the points α and β are from the relations: $\bar{\theta}\alpha - c = \alpha$ and $\underline{\theta}\beta - c = \beta$ which give $\alpha = c/(\bar{\theta} - 1)$ and $\beta = c/(\underline{\theta} - 1)$. Recall that Proposition 1 of section II says that $V(x) = 0$ for $x \leq \alpha$, $V(x) = 1$ for $x \geq \beta$ and $V(x) \in (0,1)$ for $x \in (\alpha, \beta)$. We also have $V(x)$ strictly increasing on $[\alpha, \beta]$ by Proposition 2. Further, the functional equation of $V(x)$, given by (3), takes the particular form:

$$V(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} V(\theta x - c)\, d\theta \tag{7}$$

which, after a change of variables, can be written as:

$$xV(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \int_{\underline{\theta}x - c}^{\bar{\theta}x - c} V(t)\, dt \tag{8}$$

By following the methodology used in section IV, one can show from (8) that V is, in fact, thrice differentiable on $[\alpha, \beta]$.

Define the points Z_1 and Z_2 in (α, β) as follows:

$$Z_1 = [c\underline{\theta}/\bar{\theta}(\underline{\theta} - 1)] \quad \text{and} \quad Z_2 = [c\bar{\theta}/\underline{\theta}(\bar{\theta} - 1)] \tag{9}$$

Then $Z_1 \leq Z_2$ if and only if

$$\underline{\theta}\bar{\theta} \geq \underline{\theta} + \bar{\theta} \tag{10}$$

This condition, as we shall see, is crucial to the characterization of the shape of V on the entire interval $[\alpha, \beta]$. We begin by stating the following lemma:

Lemma 1: *The survival probability function, V, satisfies $V'(x) > 0$ and $V''(x) < 0$ for x in $[Z_1, \beta)$.*

Proof: For $x \in [Z_1, \beta)$, $\bar{\theta}x - c \geq \beta$. Equation (8) then yields

$$xV(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \left[\int_{\underline{\theta}x - c}^{\beta} V(t)\, dt + \int_{\beta}^{\bar{\theta}x - c} V(t)\, dt \right]$$

But $V(t) = 1$ for $t \geq \beta$. Hence,

$$xV(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \left[\int_{\underline{\theta}x - c}^{\beta} V(t)\, dt + (\bar{\theta}x - c - \beta) \right] \tag{11}$$

Differentiating (11), we have

$$xV'(x) + V(x) = -V(\underline{\theta}x - c)[\underline{\theta}/(\bar{\theta} - \underline{\theta})] + [\bar{\theta}/(\bar{\theta} - \underline{\theta})] \tag{12}$$

For x in (α, β), $(\underline{\theta}x - c) < x$ and $V(\underline{\theta}x - c) < V(x)$ by Proposition 2. Thus, (12) yields

$$xV'(x) > -V(x)\{1 + [\underline{\theta}/(\bar{\theta} - \underline{\theta})]\} + [\bar{\theta}/(\bar{\theta} - \underline{\theta})] = [1 - V(x)][\bar{\theta}/(\bar{\theta} - \underline{\theta})] > 0$$

by using Proposition 1. This establishes $V'(x) > 0$ for x in $[Z_1, \beta)$. Differentiating (12) we have

$$xV''(x) = -[\underline{\theta}^2/(\bar{\theta} - \underline{\theta})]V'(\underline{\theta}x - c) - 2V'(x)$$

Since x lies in $[Z_1, \beta]$, $V'(x) > 0$; also, $V'(\underline{\theta}x - c) \geq 0$. Hence $V''(x) < 0$ for $x \in [Z_1, \beta)$. //

We now proceed to show that in the range (α, Z_1), there exists a unique x^* such that $V''(x^*) = 0$ and $V''(\bullet) > 0$ on (α, x^*), $V''(\bullet) < 0$ on (x^*, β). In order to do this, we assume (10), a condition which, we believe, is not too restrictive; for example, it is always satisfied if the technology is "sufficiently productive" (specifically, if $\underline{\theta} > 2$).

Theorem 3:
If $\underline{\theta}\bar{\theta} \geq \underline{\theta} + \bar{\theta}$, then $V(x)$ is S-shaped on $[\alpha, \beta]$. That is, there exists a unique x^*, where $\alpha < x^* < Z_1$, such that $V''(x^*) = 0$, $V''(\bullet) > 0$ on (α, x^*), $V''(\bullet) < 0$ on (x^*, β).

Proof: We know that V is twice continuously differentiable on $[\alpha, \beta]$, and $V'(\alpha) = 0 = V'(\beta)$. Hence, by Rolle's theorem (see Bartle (1976), p. 196) there exists some $y \in (\alpha, \beta)$ such that $V''(y) = 0$.

Define $x^* = \sup\{x \in (\alpha, \beta)\colon V''(x) = 0\}$. Since $V'' < 0$ on $[Z_1, \beta)$, it follows that $x^* \in (\alpha, Z_1)$. Also, $V''(x^*) = 0$ and $V'' < 0$ on (x^*, β) by continuity of V''. All that remains to be shown is that $V'' > 0$ on (α, x^*).

Define the sequence of points $(x_t)_{t=0}^{\infty}$ by the recursion:

$$x_0 = x^*, \bar{\theta}x_{t+1} - c = x_t \quad \text{for } t \geq 0$$

The following properties of $\{x_t\}_{t=0}^{\infty}$ are worth noting: (i) $x_t > \alpha$ for $t \geq 0$; (ii) $x_{t+1} < x_t$ for $t \geq 0$; (iii) the sequence $\{x_t\}_{t=0}^{\infty}$ converges to α. To see (i), note first that $x_0 = x^* > \alpha$. Further if $x_t > \alpha$ for some $t \geq 0$, then $\bar{\theta}x_{t+1} - c = x_t > \alpha = \bar{\theta}\alpha - c$, which yields $x_{t+1} > \alpha$. Thus (i) follows by induction. Using (i), we get for $t \geq 0$, $x_t = \bar{\theta}x_{t+1} - c > x_{t+1}$, which establishes (ii). Using (i) and (ii), we know that x_t converges to some $\hat{x} \geq \alpha$. Then since $\bar{\theta}x_{t+1} - c = x_t$, we get $\bar{\theta}\hat{x} - c = \hat{x}$, which implies $\hat{x} = \alpha$. This establishes (iii).

Next, observe that the functional equation (8) takes the form

$$xV(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \int_{\alpha}^{\bar{\theta}x - c} V(t)\,dt \tag{13}$$

for x in (α, Z_1). This is because $x \in (\alpha, Z_1)$ implies $x < Z_1 < Z_2$ which in turn implies $\underline{\theta}x - c \leq \alpha$. Note that $V(t) = 0$ for $t \leq \alpha$. Differentiating (13), we obtain

$$xV'(x) + V(x) = [\bar{\theta}/(\bar{\theta} - \underline{\theta})]V(\bar{\theta}x - c)$$

Differentiating this equation, we obtain the fundamental differential equation for

$x \in (\alpha, Z_1)$:

$$xV''(x) + 2V'(x) = [\bar{\theta}^2/(\bar{\theta} - \underline{\theta})]V'(\bar{\theta}x - c) \tag{14}$$

The proof from this point is divided into four main steps.

Step 1: There exists $\varepsilon > 0$ such that $V''(\bullet) > 0$ on $[x^* - \varepsilon, x^*)$.

Differentiating (14) (we know V is thrice differentiable for the case considered in this section), we obtain

$$xV'''(x) + 3V''(x) = [\bar{\theta}^3/(\bar{\theta} - \underline{\theta})]V'(\bar{\theta}x - c) \tag{15}$$

Evaluating (15) at $x = x^*$ and noting that $V''(x^*) = 0$, we get

$$x^* V'''(x^*) = [\bar{\theta}^3/(\bar{\theta} - \underline{\theta})]V''(\bar{\theta}x^* - c) \tag{16}$$

But $(\bar{\theta}x^* - c)$ lies in (x^*, β) where $V'' < 0$. Hence $V'''(x^*) < 0$. Therefore, there exists $\varepsilon > 0$ such that $V'' > 0$ on $[x^* - \varepsilon, x^*)$.

Step 2. $V'' > 0$ on $[x_1, x^*)$.

If $x_1 \geq x^* - \varepsilon$, then we are done. So consider the case where $x_1 < x^* - \varepsilon$ and suppose contrary to our claim that there exists $y \in [x_1, x^* - \varepsilon)$ such that $V''(y) = 0$. Let $\hat{y} = \sup\{y \in [x_1, x^* - \varepsilon): V''(y) = 0\}$. It follows that $V''(\hat{y}) = 0$ and $\hat{y} \in [x_1, x^* - \varepsilon)$, and $V'' > 0$ on $(\hat{y}, x^*)$. Evaluating (14) at $\hat{y}$ and x^* we have:

$$2V'(\hat{y}) = [\bar{\theta}^2/(\bar{\theta} - \underline{\theta})]V'(\bar{\theta}\hat{y} - c) \tag{17}$$

$$2V'(x^*) = [\bar{\theta}^2/(\bar{\theta} - \underline{\theta})]V'(\bar{\theta}x^* - c) \tag{18}$$

Since $V'' > 0$ on $(\hat{y}, x^*)$, $V'(\hat{y}) < V'(x^*)$. Then, (17), (18) yield

$$V'(\bar{\theta}\hat{y} - c) < V'(\bar{\theta}x^* - c) \tag{19}$$

Now, $\hat{y} \in [x_1, x^* - \varepsilon)$ implies $(\bar{\theta}\hat{y} - c)$ lies in (x^*, β) and so does $(\bar{\theta}x^* - c)$. We know that $V'' < 0$ on (x^*, β). Since $\hat{y} < x^*$, we have $\bar{\theta}\hat{y} - c < \bar{\theta}x^* - c$ and hence $V'(\bar{\theta}\hat{y} - c) > V'(\bar{\theta}x^* - c)$, which contradicts (19).

Step 3. $V''(\bullet) > 0$ on $[x_2, x_1)$.

Suppose on the contrary, there exists $z \in [x_2, x_1)$ such that $V''(z) = 0$. Let $\hat{z} = \sup\{z \in [x_2, x_1): V''(z) = 0\}$. Hence, $V''(\hat{z}) = 0$ and $V'' > 0$ on $(\hat{z}, x_1)$. Since $V''(x_1) > 0$, $\hat{z} < x_1$. Evaluating (14) at $x = \hat{z}$, and using $V''(\hat{z}) = 0$ we must have:

$$2V'(\hat{z}) = [\bar{\theta}^2/(\bar{\theta} - \underline{\theta})]V'(\bar{\theta}\hat{z} - c) \tag{20}$$

But $\bar{\theta}\underline{\theta} \geq \bar{\theta} + \underline{\theta} > 2\underline{\theta}$ implies $\bar{\theta} > 2$. Hence $[\bar{\theta}^2/(\bar{\theta} - \underline{\theta})] > 2$. Further, $\hat{z} \in [x_2, x_1)$ implies $(\bar{\theta}\hat{z} - c) \in [x_1, x^*)$. Hence $V'' > 0$ on $(\hat{z}, \bar{\theta}\hat{z} - c)$ so that $V'(\hat{z}) < V'(\bar{\theta}\hat{z} - c)$. In other words, $[\bar{\theta}^2/(\bar{\theta} - \underline{\theta})]V'(\bar{\theta}\hat{z} - c) > 2V'(\hat{z})$, contradicting (20).

Step 4. If $V'' > 0$ on $[x_t, x_{t-1})$, then $V'' > 0$ on $[x_{t+1}, x_t)$, $t \geq 1$.

The proof is identical to Step 3 and hence omitted. Thus by induction, $V'' > 0$ on $[x_t, x_{t-1})$ for all $t \geq 1$. Since $x_t \to \alpha$ as $t \to \infty$, we have $V'' > 0$ on (α, x^*). //

V.b. The case of a nonlinear technology with uniformly distributed multiplicative shocks

In this subsection we maintain (A.1) and (A.2) and assume that $h(x, \theta) = \theta f(x)$, where f is an increasing, strictly concave function (on the relevant range of input stocks) and the random shocks enter multiplicatively. As before, we continue to assume that $g(\theta)$ is the uniform density on $[\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} < \bar{\theta} < \infty$. In order to avoid any ambiguity, we state below the entire set of conditions that will be imposed on the function, f.

(C.1) *f is continuous on $\Re_+$ and thrice differentiable on $\Re_{++}$.*
(C.2) *$f(x) = 0$ for $x \leq 0$.*
(C.3) *For any $\theta \in [\underline{\theta}, \bar{\theta}]$, there exists $x(\theta) > 0$ such that $\theta f(x(\theta)) - c = x(\theta)$ and $\theta f(x) - c < x$ for all $x \in (0, x(\theta))$.*
(C.4) *f is strictly increasing on $[\alpha, \beta]$, (with $f' > 0$ on $[\alpha, \beta]$), where $\alpha = x(\bar{\theta})$ and $\beta = x(\underline{\theta})$.*
(C.5) *f is strictly concave on $[\alpha, \beta]$ (with $f'' < 0$ on $[\alpha, \beta]$).*

As in section V.a, define the points Z_1 and Z_2 in (α, β) by $\bar{\theta} f(Z_1) - c = \beta$ and $\underline{\theta} f(Z_2) - c = \alpha$.

First of all note that the functional equation (3) takes the following form for the technology considered in this section:

$$V(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} V(\theta f(x) - c) d\theta$$

which after change of variables can be written as:

$$f(x) V(x) = \frac{1}{\bar{\theta} - \underline{\theta}} \int_{\underline{\theta} f(x) - c}^{\bar{\theta} f(x) - c} V(t) dt$$

By following the method outlined in section IV and using (C.1), one can show that V is thrice differentiable on (α, β).

As in Theorem 3, in order to establish that V is S-shaped, we need to ensure that $Z_1 \leq Z_2$. We *assume* this directly:

(C.6) $Z_1 \leq Z_2$.

Lastly, we need to make a technical assumption which appears crucial to our method of proof. Define $q: \Re_{++} \to \Re$ by

$$q(x) = \frac{1}{f'(x)} \left[2 - \frac{f(x) f''(x)}{(f'(x))^2} \right] \quad \text{for } x > 0$$

It is obvious that, under (C.1)–(C.5), $q(x) > 0$ on $[\alpha, \beta]$. We assume,

(C.7) $q'(x) \geq 0$ on $[\alpha, \beta]$.

Assumption (C.7) is satisfied for the well-known parametric example, $f(x) = x^{\mu}$, where $0 < \mu < 1$.

By employing methods essentially analogous to those used in proving Theorem 3, we can establish the following result. [For a complete proof, the interested reader is referred to Mitra and Roy (1990a)].

Theorem 4:
The survival probability function, V, is s-shaped on $[\alpha, \beta]$. *That is, there exists a unique* $x^* \in (\alpha, Z_1)$ *such that* $V''(x^*) = 0$, $V'' > 0$ *on* (α, x^*) *and* $V'' < 0$ *on* (x^*, β).

VI. Implications of the *S*-shape of the survival probability function: an example of aid allocation

Suppose there are n economic agents each characterized by a certain level of initial wealth x_i, $i = 1, 2, \ldots, n$. We assume that each agent faces the kind of production uncertainty and survival problem as outlined in the earlier sections. In particular, we assume that the survival probability function $V(x)$ faced by each economic agent is identical. Further $x_i \in (\alpha, \beta)$, $i = 1, 2, \ldots, n$ where α and β are defined as in Section II. Let $V(x)$ be strictly increasing, twice continuously differentiable and "S-shaped" on $[\alpha, \beta]$ in the sense that there exists $x^* \in (\alpha, \beta)$ such that $V'' > 0$ on (α, x^*) and $V'' < 0$ on (x^*, β).

A total budget $b > 0$ is available for allocation among the n agents so as to enhance their probability of survival. Each agent receives as aid $a_i \geq 0$. The problem is to allocate the budget among the n agents so as to maximize the sum of increments in their probability of survival. Formally:

$$\left.\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{n} [V(x_i + a_i) - V(x_i)] \\ &\text{Subject to} \quad \sum_{i=1}^{n} a_i \leq b \\ &\qquad\qquad\quad a_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}\right\} \qquad \text{(M)}$$

While existence of an optimal aid vector (that is a vector $a^* = (a_1^*, \ldots, a_n^*)$ which solves problem (M)) is immediate, it is difficult to ensure uniqueness of the optimal aid vector for all $b > 0$. Nevertheless, certain interesting possibilities can be indicated.

To begin with, note that the first-order necessary conditions for maximization imply that if for some $b > 0$ an aid vector a^* is optimal and satisfies $a_i^* > 0$, $a_j^* > 0$, $i \neq j$, then

$$V'(x_i + a_i^*) = V'(x_j + a_j^*) \tag{21}$$

And, if $a_i^* > 0$, $a_j^* = 0$, then

$$V'(x_i + a_i^*) \geq V'(x_j) \tag{21'}$$

The first important fact worth noting is that it is by no means necessary that the poorest agent receives the largest aid. In fact no such monotonicity with respect to initial wealth may hold because of the S-shape of $V(x)$. For example, consider the case where $n = 2$, $x_1 < x^* < x_2$ (where $V''(x^*) = 0$), $V'(x_2) > V'(x_1)$. Let $\hat{b}$ be defined by $\hat{b} = \inf\{b \geq 0\colon\ b = a_1 + a_2,\ a_1 \geq 0,\ a_2 \geq 0;\ V'(x_1 + a_1) = V'(x_2 + a_2)\}$. Then $\hat{b} > 0$, and given (21), (21'), for $b \in (0, \hat{b})$, there is a unique optimal aid vector and it awards the entire available budget as aid to agent 2 and none to agent 1; that is, $a_1^* = 0$, $a_2^* = b$. Also, it can be shown that if $x_i < x_j \leq x^*$, then the lowest level of b

for which positive aid is given to i (under some optimal aid vector) is greater than that for j. [These and other results are discussed in Mitra and Roy (1990b)].

One cannot discuss the monotonicity of the optimal aid of a particular agent as a *function* of b, because of possible non-uniqueness of solutions. However, indications are that a selection from the optimal aid correspondence could be cyclical.

VII. Sensitivity of the probability of survival to changes in the distribution of the random shocks

Consider the probability of survival in the process (1) defined in section II, under two alternative sequences of random shocks $\{\hat{\theta}_t\}_{t=1}^{\infty}$ and $\{\tilde{\theta}_t\}_{t=1}^{\infty}$. Let the stochastic process of inputs for a given level of subsistence consumption c from some initial stock y corresponding to the sequence $\{\tilde{\theta}_t\}_{t=1}^{\infty}$ be denoted by $\{\tilde{X}_t(y)\}_{t=0}^{\infty}$. More formally:

$$\hat{X}_0(y) = y, \hat{X}_t(y) = h(\hat{X}_{t-1}(y), \hat{\theta}_t) - c, \quad t \geq 1 \tag{22}$$

$$\tilde{X}_0(y) = y, \tilde{X}_t(y) = h(\tilde{X}_{t-1}(y), \tilde{\theta}_t) - c, \quad t \geq 1 \tag{23}$$

We impose a very weak set of restrictions on the return function and the distribution of the random shocks, which can be stated as follows:

(D.1) $h(x, \theta) = 0$ *for* $x \leq 0$; h *is increasing in* x *and* θ.
(D.2) $\{\hat{\theta}_t\}_{t=1}^{\infty}$ *and* $\{\tilde{\theta}_t\}_{t=1}^{\infty}$ *are sequences of independent random variables.*

Note that we do not require all the $\hat{\theta}_t$'s or all the $\tilde{\theta}_t$'s to be identically distributed.

Let $\hat{P}(y)$ and $\tilde{P}(y)$ denote the probability of survival in the processes (22) and (23) respectively. The probability of survival for t periods is denoted by $\hat{P}^t(y)$ and $\tilde{P}^t(y)$, respectively. That is, $\hat{P}^t(y) =$ Probability $[\hat{X}_\tau(y) > 0$ for $\tau = 1, \ldots, t]$; $\tilde{P}^t(y) =$ Probability $[\tilde{X}_\tau(y) > 0$ for $\tau = 1, \ldots, t]$.

We say that a random variable X is *stochastically larger than* Y, denoted by $X \geq \text{st } Y$ if $P[X > a] \geq P[Y > a]$ for all $a \in \Re$. The following result is well known (see Ross (1983, p. 256)) and useful for our analysis.

Proposition 3:
If $X_1, \ldots, X_n$ *are independent,* $Y_1, \ldots, Y_n$ *are independent and* $X_i \geq \text{st } Y_i$ *for all* $i = 1, \ldots, n$, *then for any increasing* f, $f(X_1, \ldots, X_n) \geq \text{st } f(Y_1, \ldots, Y_n)$.

The following result is obtained as an easy application of Proposition 3.

Lemma 3:
If for all $t \geq 1$, $\hat{\theta}_t \geq \text{st } \tilde{\theta}_t$, *then under* (D.1) *and* (D.2), $\hat{X}_t(y) \geq \text{st } \tilde{X}_t(y)$, *for every* y.

Proof: We have $\hat{X}_1(y) = h(y, \hat{\theta}_1) - c$ and $\tilde{X}_1(y) = h(y, \tilde{\theta}_1) - c$. Now, $\hat{\theta}_1 \geq \text{st } \tilde{\theta}_1$ implies, from Proposition 3 (for $n = 1$), that $\hat{X}_1 \geq \text{st } \tilde{X}_1$. Suppose for some $t \geq 1$, $\hat{X}_t(y) \geq \text{st } \tilde{X}_t(y)$. Then, $\hat{X}_{t+1}(y) = h(\hat{X}_t(y), \hat{\theta}_{t+1}) - c$, and $\tilde{X}_{t+1}(y) = h(\tilde{X}_t(y), \tilde{\theta}_{t+1}) - c$. Note that $\hat{X}_t(y)$ and $\hat{\theta}_{t+1}$ are independent and so are $\tilde{X}_t(y)$ and $\tilde{\theta}_{t+1}$. Thus again, by Proposition 3, $\hat{X}_{t+1}(y) \geq \text{st } \tilde{X}_{t+1}(y)$. Thus, by induction, $\hat{X}_t(y) \geq \text{st } \tilde{X}_t(y)$ for all $t \geq 1$. //

Theorem 5:
If for all $t \geq 1$, $\hat{\theta}_t \geq \text{st}\, \tilde{\theta}_t$, *then under* (D.1) *and* (D.2), $\hat{P}(y) \geq \tilde{P}(y)$.

Proof: For any $t \geq 1$, $\hat{P}^t(y) = \text{Prob}[\hat{X}_t(y) > 0]$, and $\tilde{P}^t(y) = \text{Prob}[\tilde{X}_t(y) > 0]$. Further, from Lemma 3, $\hat{X}_t(y) \geq \text{st}\, \tilde{X}_t(y)$. Hence $\text{Prob}[\hat{X}_t(y) > 0] \geq \text{Prob}[\tilde{X}_t(y) > 0]$; that is $\hat{P}^t(y) \geq \tilde{P}^t(y)$. Since $\hat{P}^t(y) \downarrow \hat{P}(y)$ and $\tilde{P}^t(y) \downarrow \tilde{P}(y)$ as $t \to \infty$ we have the required result. //

One may be curious as to how the probabilities of survival between the two processes (22) and (23) compare if there is a difference in the variability or "riskiness" in the two sets of random variables – in particular, if the $\tilde{\theta}_t$'s have distributions that are mean-preserving spreads of the distribution of the $\hat{\theta}_t$'s. Consider the particular case where $\{\hat{\theta}_t\}_{t=1}^{\infty}$ is a sequence of independent and identically distributed random variables with the common distribution being uniform on some interval $[a_1, b_1] \subset (1, \infty)$. Similarly, take the $\tilde{\theta}_t$'s to be independent and identically distributed with the common distribution being uniform on some interval $[a_2, b_2] \subset (1, \infty)$.

The notion that the distribution of $\tilde{\theta}_t$'s is a mean preserving spread of that of the $\hat{\theta}_t$'s implies,

$$a_2 < a_1 < b_1 < b_2, \quad \text{and} \quad (a_1 + b_1) = (a_2 + b_2) \tag{24}$$

Let us take $h(x, \theta)$ to be linear; that is $h(x, \theta) = x\theta$ for all $x \geq 0$ and $h(x, \theta) = 0$ for $x \leq 0$. Then from the results obtained in Section II, we know there exist points $\alpha_1, \alpha_2, \beta_1$ and β_2 where $\alpha_1 = c/(b_1 - 1)$, $\beta_1 = c/(a_1 - 1)$, $\alpha_2 = c/(b_2 - 1)$, $\beta_2 = c/(a_2 - 1)$ such that: $\hat{P}(y) = 0$ for $y \leq \alpha_1$, $\hat{P}(y) = 1$ for $y \geq \beta_1$ and $0 < \hat{P}(y) < 1$ for $y \in (\alpha_1, \beta_1)$. Similarly $\tilde{P}(y) = 0$ for $y \leq \alpha_2$, $\tilde{P}(y) = 1$ for $y \geq \beta_2$ and $0 < \tilde{P}(y) < 1$ for $y \in (\alpha_2, \beta_2)$. We also know that $\hat{P}$ is strictly increasing and continuous on $[\alpha_1, \beta_1]$, $\tilde{P}$ is strictly increasing and continuous on $[\alpha_2, \beta_2]$. Lastly, (24) implies $\alpha_2 < \alpha_1 < \beta_1 < \beta_2$. Define $\xi_1 = \inf(z : \tilde{P}(z) = \hat{P}(z), z \in [\alpha_1, \beta_1]\}$, and $\xi_2 = \sup\{z : \tilde{P}(z) = \hat{P}(z), z \in [\alpha_2, \beta_2]\})$. Then $\alpha_1 < \xi_1 \leq \xi_2 < \beta_1$ and $\tilde{P}(y) > \hat{P}(y)$ for $y \in (\alpha_2, \xi_1)$ while $\hat{P}(y) > \tilde{P}(y)$ for $y \in (\xi_2, \beta_2)$.

Thus a higher variability of the distribution of the θ_t's in the sense of mean-preserving spread does not necessarily imply that the probability of survival will be higher or lower for all levels of y. In fact, as we see in the above example, for certain ranges of values for y, typically a "low range", the probability of survival is higher in a process with greater variability of random shocks. The opposite holds for some "high range" of y. This may, in part, explain some kind of "risk-loving" behavior of an agent concerned about survival at low levels of initial wealth and "risk-averse" behavior at some high levels of initial wealth. The possibility of this kind of behavior has been noted in Majumdar and Radner (1991).

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